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A METHOD OF EXAMINING A PAIR OF INTERDEPENDENT INTEGRAL EQUATIONS

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The paper deals with the method of inverting two singular integral equations of the first and second kind, respectively, possessing a definite structure. The equations as well as their solutions are obtained on the basis of analyzing a specific mixed problem of the potential theory for a quadrant.

1. Let us seek two functions, $\varphi_1(z)$ and $\chi_1(z)$ regular in the upper right quadrant, vanishig at infinity and satisfying the following conditions at the boundary rays:

$$\varkappa \varphi_1(t) + \overline{\chi_1(t)} = f_1(t), \quad t = x, \quad 0 \leqslant x < \infty \tag{1.1}$$

$$\varphi_1(t) + \overline{\chi_1(t)} = f_2(t); \quad t = iy, \quad 0 \leqslant y < \infty \tag{1.2}$$

Generally speaking, $\varkappa$ is a complex parameter and the specified functions $f_1(t)$ and $f_2(t)$ satisfy the Hölder's condition and are of the order $O(1/t)$ at infinity. In what follows we shall assume, without loss of generality, that $f_2(t) = 0$. We arrive at this case by subtracting from the solution which is being sought, the particular solution for the right semi-plane with the condition (1.2) holding along its whole boundary (and in particular, when $f_2(t)$ is zero on the negative half of the ordinates).

Let us introduce the auxilliary function $\omega(t)$ on the ray $(0 \leqslant x < \infty)$

$$A\varphi_1(t) - \overline{\chi_1(t)} = 2\omega(t), \qquad 0 \leqslant t < \infty \tag{1.3}$$

where A is a certain complex constant. Adding and subtracting (1.1) and (1.3) term by term, we obtain

$$\varphi_1(t) = \frac{1}{A+\varkappa}[2\omega(t) + f_1(t)] \tag{1.4}$$

$$\chi_1(t) = \frac{1}{\bar{A}+\bar{\varkappa}}[-2\bar{\varkappa}\,\overline{\omega(t)} + \bar{A}\overline{f_1(t)}]$$

Let us now define new functions $\varphi(z)$ and $\chi(z)$ regular in the upper right quadrant

$$\varphi(z) = \varphi_1(z) - \frac{1}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t-z}dt - \frac{1}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t-z}dt \tag{1.5}$$

$$\chi(z) = \chi_1(z) + \frac{\bar{\varkappa}}{\bar{A}+\bar{\varkappa}}\frac{1}{\pi i}\int_0^\infty \frac{\overline{\omega(t)}}{t-z}dt - \frac{\bar{A}}{\bar{A}+\bar{\varkappa}}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t-z}dt$$

The above functions are analytically continued into the lower right quadrant across the positive semi-axis using the formulas

$$\varphi(z) = \frac{1}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t-z}dt - \frac{1}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t-z}dt \tag{1.6}$$

$$\chi(z) = \frac{\bar{\varkappa}}{\bar{A}+\bar{\varkappa}}\frac{1}{\pi i}\int_0^\infty \frac{\overline{\omega(t)}}{t-z}dt - \frac{\bar{A}}{\bar{A}+\bar{\varkappa}}\frac{1}{\pi i}\int_0^\infty \frac{\overline{f_1(t)}}{t-z}dt$$

Let us insert the expressions for $\varphi_1(z)$ and $\chi_1(z)$ given by (1.5), into (1.1) and (1.3). Then the Sokhotskii-Plemelj formula yields, after simple manipulations, the following relations along the semi-axis $x \geqslant 0$:

$$\varkappa\varphi(t_0) + \overline{\chi(t_0)} + \frac{2\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t-t_0}dt = \frac{f_1(t_0)}{2} + \frac{A-\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt \tag{1.7}$$

$$A\varphi(t_0) - \overline{\chi(t_0)} + \frac{A-\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t-t_0}dt + \frac{A}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt = \omega(t_0) \tag{1.8}$$

The relations (1.5) and (1.6) lead to the following condition on the ordinate for the functions $\varphi(z)$ and $\chi(z)$ regular in the right semi-plane:

$$\varphi(t_0) + \overline{\chi(t_0)} = -\frac{1}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t-t_0}dt - \frac{\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t+t_0}dt - \tag{1.9}$$

$$\frac{1}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt + \frac{A}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt$$

Addition to this equation of its conjugate yields, after elementary manipulations,

$$\varphi(z) = \frac{\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t+z}dt + \frac{A}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t+z}dt \tag{1.10}$$

$$\chi(z) = \frac{1}{\bar{A}+\bar{\varkappa}}\frac{1}{\pi i}\int_0^\infty \frac{\overline{\omega(t)}}{t+z}dt + \frac{1}{\bar{A}+\bar{\varkappa}}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t+z}dt$$

Next we substitute the above expressions for $\varphi(z)$ and $\chi(z)$ first into (1.7), then into (1.8). This yields a pair of singular integral equations of the first and second kind

$$\frac{2\varkappa}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \omega(t)\left(\frac{1}{t-t_0} - \frac{\varkappa^2+1}{2\varkappa}\frac{1}{t+t_0}\right)dt = \tag{1.11}$$

$$\frac{f_1(t_0)}{2} + \frac{A-\varkappa}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt + \frac{1-A\varkappa}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t+t_0}dt$$

$$\omega(t_0) - \frac{A-\varkappa}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{\omega(t)}{t-t_0}dt - \frac{1-Ax}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{\omega(t)}{t+t_0}dt = \tag{1.12}$$

$$\frac{A}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt + \frac{1-A^2}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t+t_0}dt$$

Let us set in the last equation $A = 1/\varkappa$ and denote the auxilliary function corresponding to this particular value of the parameter A, by $\delta(t)$. Equation (1.12) becomes considerably simplified for this value and degenerates into the Carleman's equation

$$\delta(t_0) + \frac{\Delta}{\pi i}\int_0^\infty \frac{\delta(t)}{t - t_0}\,dt = g(t_0), \qquad \Delta = \frac{\varkappa^2 - 1}{\varkappa^2 + 1} \tag{1.13}$$

$$g(t_0) = \frac{1}{1 + \varkappa^2}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t - t_0}\,dt + \frac{1}{2\pi i\varkappa}\int_0^\infty \frac{f_1(t)}{t + t_0}\,dt \tag{1.14}$$

We note that when $\varkappa = \pm 1$, the singular integral in the left-hand side of (1.13) vanishes and we obtain

$$\delta(t) = \frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t - t_0}\,dt \pm \frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t + t_0}\,dt$$

With $\delta(t)$ known, we turn to the relations (1.4) to find the solutions of the simplest fundamental and mixed problems for the quadrant, in their closed form

$$\varphi_1(z) = \pm\frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t - z}\,dt + \frac{1}{2\pi i}\int_0^\infty \frac{f_1(t)}{t + z}\,dt \tag{1.15}$$

$$\chi_1(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\overline{f_1(t)}}{t - z}\,dt \pm \frac{1}{2\pi i}\int_0^\infty \frac{\overline{f_1(t)}}{t + z}\,dt$$

We write the solution of (1.13) (nontrivial solution of the homogeneous equation is omitted; it can be easily derived if needed) [1, 2]

$$\delta(t_0) = \frac{1}{1 - \Delta^2}g(t_0) - \frac{\Delta}{1 - \Delta^2}\frac{1}{\pi i t_0^{\mu}}\int_0^\infty \frac{t^{\mu}g(t)}{t - t_0}\,dt, \qquad \mu = \frac{1}{2\pi i}\ln\frac{1 - \Delta}{1 + \Delta} \tag{1.16}$$

Here the argument of the logarithm must be chosen such, that $-1 < \mathrm{Re}\,\mu < 1$. Only then the integral in the right-hand side will remain meaningful (for any $t_0 \neq 0$).

We note that when the quantity $\mathrm{Re}\,\mu$ is positive, the density $\delta(t)$ becomes easier to investigate. The case when $1 - \Delta^2$ vanishes identically is omitted, as it represents a particular case with no direct bearing on the problem in question. We use $\delta(t)$ obtained for $A = 1/\varkappa$) to find from (1.1) and (1.3) the functions $\varphi_1(t)$ and $\chi_1(t)$ on the real ray, and we continue them analytically into the quadrant S. Incidentally, the expressions obtained for these functions are very simple. Indeed, substituting $g(t)$ from (1.14) into (1.16) and computing the inner integral in the multiple integral that appears in the expression, we find, after changing the order of integration, the following simpler formula for the auxilliary function to replace (1.16)

$$\omega(t_0) = -\frac{\varkappa^2 - 1}{4\varkappa^2}f_1(t_0) - \frac{\varkappa^2 + 1}{4\varkappa^2}\frac{t_0^{-\mu}}{\pi i}\int_0^\infty t^{\mu}f_1(t)\left(\frac{1}{t - t_0} + \frac{1}{t + t_0}\right)dt \tag{1.17}$$

Using the above formula we arrive at equally simple expressions for the functions to be determined

$$\varphi_1(z) = \frac{\varkappa^2 + 1}{4\varkappa^3}\frac{z^{-\mu}}{\pi i}\int_0^\infty t^{\mu}f_1(t)\left(\frac{1}{t - z} + \frac{1}{t + z}\right)dt \tag{1.18}$$

$$\chi_1(z) = \frac{\bar{\varkappa}^2+1}{4\bar{\varkappa}^2}\,\frac{z^{-\bar{\mu}}}{\pi i}\int_0^\infty t^{\bar{\mu}}\overline{f_1(t)}\left(\frac{1}{t-z}+\frac{1}{t+z}\right)dt$$

The above expressions satisfy the basic conditions at the quadrant boundaries identically (this can be confirmed by taking into account the fact that $\exp(\pi i\mu) = \varkappa^{-1}$).

2. Let us return to the Eqs. (1.11) and (1.12). Assuming that $A \neq 1/\varkappa$ and $A^2 + 1 \neq 0$, we replace the density $\omega(t)$ in these equations by $\omega^*(t)$ according to the formulas

$$\omega^*(t) = \omega(t) - \frac{A\varkappa - 1}{2(\varkappa^2+1)} f_1(t), \qquad \omega^*(t) = \omega(t) + \frac{1+A^2}{2(1-A\varkappa)} f_1(t) \tag{2.1}$$

This appreciably simplifies their right-hand sides which consequently become

$$\frac{1}{\pi i}\int_0^\infty \omega^*(t)\left[\frac{1}{t-t_0} - \frac{\varkappa^2+1}{2\varkappa}\,\frac{1}{t+t_0}\right]dt = \tag{2.2}$$

$$\frac{A+\varkappa}{4\varkappa}\left[f_1(t_0) - \frac{\varkappa^2-1}{\varkappa^2+1}\,\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}\,dt\right]$$

$$\omega^*(t_0) - \frac{1}{(A+\varkappa)\pi i}\int_0^\infty \omega^*(t)\left[\frac{A-\varkappa}{t-t_0} + \frac{1-A\varkappa}{t+t_0}\right]dt = \tag{2.3}$$

$$\frac{1+A^2}{2(1-A\varkappa)}\left[f_1(t_0) + \frac{1-A^2}{1+A^2}\int_0^\infty \frac{f_1(t)}{t-t_0}\,dt\right]$$

We note that in the first of these equations the passage to the value $A = 1/\varkappa$ when $\omega^*(t) = \omega(t)$, is still admissible.

We shall naturally attempt to solve the equation of the first kind

$$\frac{1}{\pi i}\int_0^\infty \mu(t)\left[\frac{1}{t-t_0} - \lambda\,\frac{1}{t+t_0}\right]dt = f(t_0) \tag{2.4}$$

and then the equation of the second kind

$$\mu(t_0) + \frac{1}{\pi i}\int_0^\infty \mu(t)\left[\frac{B}{t-t_0} + \frac{C}{t+t_0}\right]dt = f(t_0) \tag{2.5}$$

combining each of them with the boundary value problem (1.1), (1.2). Here λ, B and C are known, generally speaking, complex constants, and $f(t)$ is any specified function.

According to the transformation realized previously, the solutions of (1.11), (1.12) and (2.2), (2.3) depend on the relation (2.1) where $\omega(t)$, in its turn, easily links to $\delta(t)$ computed according to (1.13). On the other hand, it is obvious that the solution of the singular equation (2.5) (of a more complex structure) coincides identically with the density $\omega^*(t)$ which satisfies (2.3), provided that the coefficients B, C and free term $f(t)$ are specified, while the parameter $\varkappa$, function $f(t)$ (these two can be added to the exhaustive characteristics of the initial boundary value problem) and A, and found from the relations (*)

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*) A similar method was used in [3] while investigating singular equations of a different structure.

$$B = -\frac{A-\varkappa}{A+\varkappa},\quad C = -\frac{1-A\varkappa}{A+\varkappa}$$

$$f_1(t_0) + \frac{1-A^2}{1+A^2}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt = \frac{2(1-A\varkappa)}{1+A^2}f(t_0)$$

It is evident that the fulfilment of these conditions leads to the equations (2.3) and (2.5) becoming identical. We should also note that Eq. (2.4) of the first kind can be analyzed directly and individually. Setting in (2.2) $A = 1/\varkappa$ which is legitimate and implies the conversion of $\omega^*(t)$ into $\delta(t)$ and choosing the parametei $\varkappa$ and function $f(t)$ in Eq. (2.4) so that the conditions

$$\lambda = \frac{\varkappa^2+1}{2\varkappa},\quad f_1(t) - \frac{\varkappa^2-1}{\varkappa^2+1}\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}dt = \frac{4\varkappa^2}{1+\varkappa^2}f(t_0)$$

hold, we obtain our equation in the same form as (2.2). By virtue of this, the solution of (2.4) in general coincides with the value of the density given by (1.16) (or it will differ from it, at the most, by the nontrivial solutions of the homogeneous integral equation (1.13); exactly the same relationship exists between the solutions of comparable equations of the second kind).

3. Let us study in more detail a singular equation of the first kind

$$\frac{1}{\pi i}\int_0^\infty \omega(t)\left[\frac{1}{t-t_0} - \lambda\frac{1}{t+t_0}\right]dt = f(t_0),\qquad 0 < t_0 < \infty \tag{3.1}$$

Let its free term vanish at infinity as before, at the rate not slower than $O(1/t)$. We simplify the solving procedure by assuming the parameter λ to be real and not greater than unity, so that $\lambda = \cos\theta$, $0 \leqslant \theta \leqslant \pi$. We also assume that $\alpha = 1-\theta/\pi \leqslant 1$.

After performing the manipulations indicated we confirm that Eq. (3.1) has two types of solutions, and we denote them by $\omega(t_0)$ and $\mu(t_0)$. Their final expressions in closed form are

$$\omega(t_0) = t_0^\alpha\int_0^\infty t^{-\alpha}f(t)\left[\frac{1}{t-t_0} - \frac{1}{t+t_0}\right]dt + \tag{3.2}$$

$$t_0^{-\alpha}\int_0^\infty t^\alpha f(t)\left[\frac{1}{t-t_0} - \frac{1}{t+t_0}\right]dt$$

$$\mu(t_0) = t_0^{\frac{\theta}{\pi}}\int_0^\infty t^{-\frac{\theta}{\pi}}f(t)\left[\frac{1}{t-t_0} + \frac{1}{t+t_0}\right]dt +$$

$$t_0^{-\frac{\theta}{\pi}}\int_0^\infty t^{\frac{\theta}{\pi}}f(t)\left[\frac{1}{t-t_0} + \frac{1}{t+t_0}\right]dt$$

We note that the first of these solutions is continuous at the coordinate origin, while the second solution at this point becomes, as a rule, discontinuous. The density $\omega(t)$ vanishes at infinity, generally speaking, more slowly than could be expected, namely as $O(t^{-(1-\beta)})$. At the same time, the solution $\mu(t)$ at the distant parts of the region is of the order $O\,|\,t^{-(1+\gamma)}\,|$, where γ as well as β are positive numbers obviously smaller than unity. The validity of (3.2) can be confirmed by direct substitution of the densities into the integral equation (3.1). After some fairly cumbersome manipulations we find that the equation is satisfied identically in each case. In the course of these manipulations the following formulas for reducing the double integrals encountered to single integrals, are found useful:

$$\frac{1}{\pi i}\int_0^\infty t^{\pm\alpha}\frac{dt}{t-t_0}\frac{1}{2\pi i}\int_0^\infty t_1^{\mp\alpha} f(t_1)\frac{dt_1}{t_1-t} = \frac{1}{2} f(t_0) \pm$$

$$i\frac{\cos\pi\alpha}{\sin\pi\alpha}\frac{1}{2\pi i}\int_0^\infty (t_0^{\pm\alpha}t^{\mp\alpha}-1) f(t)\frac{dt}{t-t_0}$$

$$\frac{1}{\pi i}\int_0^\infty t^{\pm\alpha}\frac{dt}{t-t_0}\frac{1}{2\pi i}\int_0^\infty t_1^{\mp\alpha} f(t_1)\frac{dt_1}{t_1+t} =$$

$$\pm i\frac{1}{\sin\pi\alpha}\frac{1}{2\pi i}\int_0^\infty (\cos\pi\alpha t_0^{\pm\alpha}t^{\mp\alpha}-1) f(t)\frac{dt}{t+t_0}$$

$$\frac{1}{\pi i}\int_0^\infty t^{\pm\alpha}\frac{dt}{t+t_0}\frac{1}{2\pi i}\int_0^\infty t_1^{\mp\alpha} f(t_1)\frac{dt_1}{t_1-t} =$$

$$\pm i\frac{1}{\sin\pi\alpha}\frac{1}{2\pi i}\int_0^\infty (t_0^{\pm\alpha}t^{\mp\alpha}-\cos\pi\alpha) f(t)\frac{dt}{t+t_0}$$

$$\frac{1}{\pi i}\int_0^\infty t^{\pm\alpha}\frac{dt}{t+t_0}\frac{1}{2\pi i}\int_0^\infty t_1^{\mp\alpha} f(t_1)\frac{dt_1}{t_1+t} =$$

$$\pm i\frac{1}{\sin\pi\alpha}\frac{1}{2\pi i}\int_0^\infty (t_0^{\pm\alpha}t^{\mp\alpha}-1) f(t)\frac{dt}{t-t_0}$$

Let us now supplement the densities (3.2) with terms of the type $C_1 t_0^{-(1-\alpha)}$ and $C_2 t_0^{-(1-\alpha)}$, respectively. The latter terms represent the solutions of the homogeneous equation (3.1). Expressing now one of the constants C_1 and C_2 in terms of the other (depending on certain functionals related to $f(t)$), we reduce the expressions for the densities in their modified form into full agreement (although this entails fairly complicated transformations).

Note 1. The resolving formulas (3.2) remain structurally unchanged and can still be used when the integration in (3.1) and hence in (3.2) is performed along a semi-infinite line situated in the first or the fourth quadrant, or even more generally, above or below the ray $(0,\infty)$. This follows from the fact that neither Eq. (3.1) itself, nor the corresponding formuals (3.2) include quantities related to both the density $\omega(t)$ and the free term $f(t)$. The simplest way of confirming the above statement in a complicated case, is direct checking of the transformation formulas. To do this we replace the density $\omega(t)$ in (3.1) (with a modified contour of integration) consecutively by its expression (similarly modified) from any of the formulas (3.2); then it only remains to verify that the double integral obtained in this manner for any affix t_0 pertaining to the semi-infinite line becomes identically equal to the specified free term of (3.1).

Note 2. Returning to the first relation of (1.4) we now convert Eqs. (1.11) and (1.12) into a form containing the unknown function $\varphi_1(t)$ itself instead of the auxilliary function $\omega(t)$. After elementary manipulations we obtain

$$\frac{1}{\pi i}\int_0^\infty \varphi_1(t)\left(\frac{1}{t-t_0}-\frac{\varkappa^2+1}{2\varkappa}\,\frac{1}{t+t_0}\right)dt = \tag{3.3}$$

$$\frac{1}{2\varkappa}\left[f_1(t_0)+\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}\,dt-\frac{\varkappa}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}\,dt\right]$$

$$\varphi_1(t_0)-\frac{A-\varkappa}{A+\varkappa}\,\frac{1}{\pi i}\int_0^\infty \frac{\varphi_1(t)}{t-t_0}\,dt-\frac{1-A\varkappa}{A+\varkappa}\,\frac{1}{\pi i}\int_0^\infty \frac{\varphi_1(t)}{t+t_0}\,dt=$$

$$\frac{1}{A+\varkappa}\left[f_1(t_0)+\frac{1}{\pi i}\int_0^\infty \frac{f_1(t)}{t-t_0}\,dt+\frac{A}{\pi i}\int_0^\infty \frac{f_1(t)}{t+t_0}\,dt\right]$$

It is evident that the function $\varphi_1(z)$ is completely determined from the boundary value problem formulated in the beginning of this paper, consequently it must be independent of the constant A appearing as a multiplier in the secondary condition (1.3) introduced solely in order to develop a method perhaps not quite standard one of studying this problem. This obvious circumstance was reflected in the fact that the first equation of (3.3) no longer contains the constant A. Nevertheless, odd as it may be, the same constant is retained in the second equation of (3.3).

What we have just said does, naturally, enhance our interest in the fact which came to light, that the solutions of (3.1) satisfying simultaneously the initial conditions (1.1) and (1.2) are invariant with respect to the constant A. The first equation of (3.3) reflects such and only such solutions. Taking into consideration these solutions of (3.1), let us differentiate both sides of the second equation of (3.3) with respect to the parameter A. This at once yields the first equation of (3.3).

4. Let us illustrate the behavior of the formulas (3.2) by an example, assuming that the free term $f(t)=[a/(a+t)]^n$, where n is any positive integer and a is a positive constant. The example helps to distinguish the nuances in the features of the behavior of the densities of (3.1).

We have the following expansions (which converge in the circle $|z+a|<a$ and differ from each other only in the sign of the power parameter):

$$z^{\pm\alpha}=e^{\pm\pi i\alpha}a^{\pm\alpha}\sum_{\nu=0}^{\infty}(-1)^\nu C_{\pm\alpha}^{\nu}\left(\frac{z+a}{a}\right)^\nu \tag{4.1}$$

Let us write these expansions separately for the positive and for the negative parameter α and then crossmultiply them term by term. Comparing both sides of the resulting equation we obtain after some manipulations the following formulas:

$$\sum_{\nu=0}^{n} C_{\pm\alpha}^{\nu}C_{\mp\alpha}^{n-\nu}=0,\qquad n\geqslant 1 \tag{4.2}$$

Replacing formally the integral value n by $n-k$, where $n>k$ and the summation index ν by the new index $n-\nu$, we obtain

$$\sum_{\nu=k}^{n} C_{\pm\alpha}^{n-\nu} C_{\mp\alpha}^{\nu-k} = 0, \qquad n > k \tag{4.3}$$

Let us now add to (4.2) and (4.3) another pair of expansions valid near the point $z = -a$, the pair again obtained from (4.1) by replacing α by the parameter θ / π. Combining these equations and remembering that $\alpha = 1 - \theta / \pi$, we arrive at a group of relations of the form

$$C_{\alpha}^{\nu} = C_{-\theta/\pi}^{\nu} + C_{-\theta/\pi}^{\nu+1}, \quad (-1)^{\nu} C_{-\alpha}^{\nu} = \sum_{k=0}^{\nu} (-1)^{k} C_{\theta/\pi}^{k} \quad (\nu = 0, 1, 2, \ldots) \tag{4.4}$$

$$C_{\theta/\pi}^{\nu} = C_{-\alpha}^{\nu} + C_{-\alpha}^{\nu-1}, \quad (-1)^{\nu} C_{-\theta/\pi}^{\nu} = \sum_{k=0}^{\nu} (-1)^{k} C_{\alpha}^{k} \quad (\nu = 0, 1, 2, \ldots)$$

Expressions for the densities (3.2) corresponding to the free term in (3.1) already mentioned, are easily obtained. The equation

$$\frac{1}{2\pi i}\int_0^\infty t^{\pm\alpha}\left(\frac{a}{a+t}\right)^n \frac{dt}{t-z} = \frac{1}{1-e^{\pm 2\pi i\alpha}}\left[z^{\pm\alpha}\left(\frac{a}{a+z}\right)^n - \right. \tag{4.5}$$

$$\left. e^{\pm\pi i\alpha} a^{\pm\alpha} \sum_{\nu=1}^{n} (-1)^{n-\nu} C_{\pm\alpha}^{n-\nu}\left(\frac{a}{a+z}\right)^{\nu}\right], \qquad \operatorname{Im} z > 0$$

yields, in the obvious manner,

$$\frac{1}{2\pi i}\int_0^\infty t^{\pm\alpha}\left(\frac{a}{a+t}\right)^n \frac{dt}{t-t_0} =$$

$$\pm\frac{i}{2\sin\pi\alpha}\left\{\cos\pi\alpha\, t_0^{\pm\alpha}\left(\frac{a}{a+t_0}\right)^n - a^{\pm\alpha}\sum_{\nu=1}^{n} (-1)^{n-\nu} C_{\pm\alpha}^{n-\nu}\left(\frac{a}{a+t_0}\right)^{\nu}\right\}$$

$$\frac{1}{2\pi i}\int_0^\infty t^{\pm\alpha}\left(\frac{a}{a+t}\right)^n \frac{dt}{t+t_0} =$$

$$\pm\frac{i}{2\sin\pi\alpha}\left\{t_0^{\pm\alpha}\left(\frac{a}{a-t_0}\right)^n - a^{\pm\alpha}\sum_{\nu=1}^{n} (-1)^{n-\nu} C_{\pm\alpha}^{n-\nu}\left(\frac{a}{a-t_0}\right)^{\nu}\right\}$$

We now add to these relations another pair of relations differing from the latter in that the parameter α is replaced by θ / π. Using all these relations as well as the formulas (3.2), we obtain the required quantities

$$\omega(t_0) = \frac{i}{2\sin\pi\alpha}\left\{\left(\frac{t_0}{a}\right)^{\alpha}\sum_{\nu=1}^{n} (-1)^{n-\nu} C_{-\alpha}^{n-\nu}\left[\left(\frac{a}{a+t_0}\right)^{\nu} - \left(\frac{a}{a-t_0}\right)^{\nu}\right] - \right. \tag{4.6}$$

$$\left. \left(\frac{t_0}{a}\right)^{-\alpha}\sum_{\nu=1}^{n} (-1)^{n-\nu} C_{\alpha}^{n-\nu}\left[\left(\frac{a}{a+t_0}\right)^{\nu} - \left(\frac{a}{a-t_0}\right)^{\nu}\right]\right\}$$

$$\mu(t_0) = \frac{i}{2\sin\theta}\left\{\left(\frac{t_0}{a}\right)^{\frac{\theta}{\pi}} \sum_{\nu=1}^{n} (-1)^{n-\nu} C_{-\theta/\pi}^{n-\nu}\left[\left(\frac{a}{a+t_0}\right)^{\nu} + \left(\frac{a}{a-t_0}\right)^{\nu}\right] - \right. \tag{4.7}$$
$$\left.\left(\frac{t_0}{a}\right)^{-\frac{\theta}{\pi}} \sum_{\nu=1}^{n} (-1)^{n-\nu} C_{\theta/\pi}^{n-\nu}\left[\left(\frac{a}{a+t_0}\right)^{\nu} + \left(\frac{a}{a-t_0}\right)^{\nu}\right]\right\}$$

Let us insert the expressions (4.6) and (4.7) for the densities into the singular equation (3.1) with the free term $f(t) = [a / (a + t)]^n$. Using the relations(4.2) – (4.4) and performing a number of manipulations we conclude that these densities are indeed the solutions of (3.1) when the right-hand side has the given particular form.

It is evident that the solution (4.6) is bounded and, that it also vanishes at the point $t = 0$, while the solution (4.7) becomes discontinuous at the same point. At some distance from the origin of the ray the expansion (4.6) consists of a set of terms containing $t^{-(2n-1)\pm\alpha}$ $(n = 1, 2, \ldots)$ as multipliers. The term containing $t^{-(1-\alpha)}$ is absent from the analogous expansion for the density $\mu(t)$. We see that the solutions of the type (4.6) and (4.7) differ from each other qualitatively. Nevertheless, the solution (4.7) with a discontinuity at the coordinate origin in which the constant C is fixed in an appropriate manner and which additionally contains the nontrivial solution of the homogeneous equation (3.1), coincides completely with the solution (4.6) which is bounded everywhere. This conclusion can be easily reached by bringing in the auxilliary relations (4.2) and (4.3). It may seem that the properties of the pair of solutions listed above are stipulated solely by a skilful rendering of the free term. Nevertheless, it must be assumed that these properties are typical and they reflect the interdependence of the solutions of the type (3.2) with an arbitrary free term $f(t)$.

Thus, if the density $\omega(t)$ is bounded at the point $t = 0$, this predetermines its behavior at infinity. We may find that this is inconsistent with the necessity of ensuring the required order of the density decrease near the point at infinity. For this reason the introduction into (4.6) which is finite at $t = 0$, of a term unbounded at this point and satisfying the homogeneous equation (3.1), may help to correct the behavior of the density under determination in accordance with the conditions of the problem.

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